The Cauchy-Schwarz Inequality and the Triangle Inequality

The Cauchy-Schwarz inequality and the triangle inequality are important techical inequalities that have widespread applications, both theoretical and practical. (In fact, as you will see below, the Cauchy Schwarz Inequality is crucial for proving the Triangle Inequality.)

Theorem 1 (The Cauchy-Schwarz Inequality) *For any vectors* $\vec{u}$ *and* $\vec{v}$*, both in* $\mathbf{R}^2$ *or both in* $\mathbf{R}^3$*,*

$$|\vec{u}\cdot\vec{v}| \quad \le \quad \|\vec{u}\|\,\|\vec{v}\|. \tag{1}$$

Proof. If $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$, then both sides of (1) equal zero; since $0\le 0$, (1) is true.

On the other hand, suppose that $\vec{u}\neq\vec{0}$ and that $\vec{v}\neq\vec{0}$. Then, as we have seen,

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\,\cos(\theta), \tag{2}$$

where θ is the angle between the vectors. Taking absolute values on both sides of (2) then gives

$$\begin{aligned} |\vec{u}\cdot\vec{v}| &= \|\vec{u}\|\,\|\vec{v}\|\,|\cos(\theta)| \\ \big(\text{Because } |\cos(\theta)|\le 1 \text{ and } \|\vec{u}\|\|\vec{v}\|\ge 0 \ \longrightarrow\big) \quad &\le \|\vec{u}\|\,\|\vec{v}\|. \ \blacksquare \end{aligned}$$

Exercise 1. When will the inequality actually be an equality?

Theorem 2 (Triangle Inequality) *For any vectors* $\vec{u}$ *and* $\vec{v}$*, both in* $\mathbf{R}^2$ *or both in* $\mathbf{R}^3$*,*

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|. \tag{3}$$

Proof. Since both sides of (3) are nonnegative, (3) is equivalent to

$$\big(\|\vec{u}+\vec{v}\|\big)^2 \le \big(\|\vec{u}\|+\|\vec{v}\|\big)^2; \tag{4}$$

this is what will be proved. First of all, one simplifies each side of (4):[1]

$$\begin{aligned} (\|\vec{u}+\vec{v}\|)^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\ &= \vec{u}\cdot(\vec{u}+\vec{v})+\vec{v}\cdot(\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2+2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2 \end{aligned} \tag{5}$$

$$\begin{aligned} (\|\vec{u}\|+\|\vec{v}\|)^2 &= (\|\vec{u}\|+\|\vec{v}\|)(\|\vec{u}\|+\|\vec{v}\|) \\ &= \|\vec{u}\|(\|\vec{u}\|+\|\vec{v}\|)+\|\vec{v}\|(\|\vec{u}\|+\|\vec{v}\|) \\ &= \|\vec{u}\|\|\vec{u}\|+\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|\|\vec{u}\|+\|\vec{v}\|\|\vec{v}\| \\ &= \|\vec{u}\|^2+2\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|^2 \end{aligned} \tag{6}$$

The proof is then completed by comparing the simplified expressions ((5) and (6)) to each other: since

$$\vec{u}\cdot\vec{v} \le |\vec{u}\cdot\vec{v}| \underset{\substack{\uparrow \\ \text{Cauchy-} \\ \text{Schwarz}}}{\le} \|\vec{u}\|\|\vec{v}\|,$$

and since expressions (5) and (6) are otherwise identical to each other,[2] expression (5) is less than or equal to expression (6). $\blacksquare$

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[1] They can be simplified in *almost* parallel fashion.

[2] Actually, *more* is being used for this step; "otherwise identical" by itself is *not* emough information (inequalities are more delicate than equations). Do you see what else is being used?